

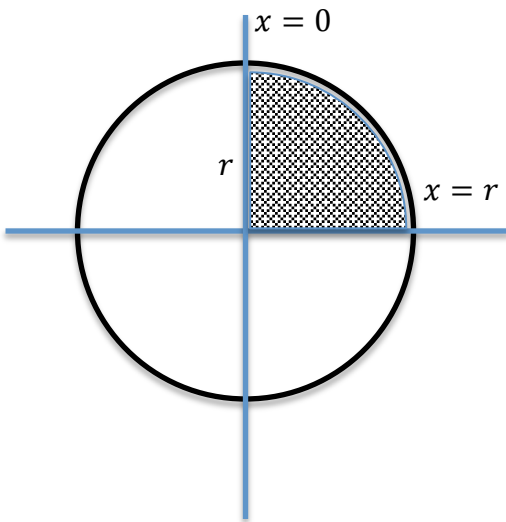
Mathematics Tutorial Series

Integral Calculus #20

Area of a Circle

A circle of radius r has equation: $x^2 + y^2 = r^2$.

We can write this as: $y = \sqrt{r^2 - x^2}$.



Area of circle =

$$4 \int_0^r \sqrt{r^2 - x^2} dx$$

Step 1: Substitute $x = ru$ so that $dx = rdu$.

Also $u = 0$ when $x = 0$ and $u = 1$ when $x = r$.

Area =

$$4 \int_0^1 \sqrt{r^2 - r^2u^2} rdu = 4r^2 \int_0^1 \sqrt{1 - u^2} du$$

Step 2: Work on $\int \sqrt{1 - u^2} du$.

First we do an integration by parts: Take $v = \sqrt{1 - u^2}$ and $du = du$. The formula is:

$$\int v du = uv - \int u dv$$

We have

$$dv = \frac{-2u}{2\sqrt{1 - u^2}} du$$

So the integration by parts goes like this:

$$\int \sqrt{1 - u^2} du = u\sqrt{1 - u^2} - \int \frac{u(-2u)}{2\sqrt{1 - u^2}} du$$

Step 3: Simplify the final term and add-and-subtract 1:

$$\begin{aligned} \int \sqrt{1 - u^2} du &= u\sqrt{1 - u^2} - \int \frac{1 - u^2 - 1}{\sqrt{1 - u^2}} du \\ &= u\sqrt{1 - u^2} - \int \frac{1 - u^2}{\sqrt{1 - u^2}} du + \int \frac{1}{\sqrt{1 - u^2}} du \end{aligned}$$

The final integral can be recognized as:

$$\int \frac{1}{\sqrt{1 - u^2}} du = \sin^{-1} u + C$$

And the other integral is simplified by noting that the numerator is the square of the denominator:

$$\int \frac{1 - u^2}{\sqrt{1 - u^2}} du = \int \sqrt{1 - u^2} du$$

Step 4: Put it together and we get:

$$\begin{aligned} \int \sqrt{1 - u^2} du &= u\sqrt{1 - u^2} - \int \sqrt{1 - u^2} du + \sin^{-1} u \\ 2 \int \sqrt{1 - u^2} du &= u\sqrt{1 - u^2} + \sin^{-1} u \end{aligned}$$

$$\int \sqrt{1-u^2} \, du = \frac{1}{2} \left(u\sqrt{1-u^2} + \sin^{-1} u \right) + C$$

Step 5: Solve the Problem

Area =

$$4r^2 \int_0^1 \sqrt{1-u^2} \, du = 4r^2 \frac{1}{2} \left(u\sqrt{1-u^2} + \sin^{-1} u \right)_0^1$$

Area of a circle of radius r =

$$2r^2(\sin^{-1}(1) - \sin^{-1}(0)) = 2r^2 \left(\frac{\pi}{2} - 0 \right) = \pi r^2$$

Step 6: Check all the steps.

To do this integral requires that you know how to do the following accurately and confidently:

1. Formulate an area problem as an integral
2. Perform a substitution in a definite integral
3. Carry out an integration by parts
4. Correctly manage the algebra for simplifying
5. Correctly manage the + and - signs
6. Recognize a standard integral - inverse sine
7. Manage the reoccurrence of the target integral
8. Correctly evaluate the anti-derivative.